MIT-WHOI Joint Program Summer Math Review **Summer 2014**

Lecture 9: Systems of Ordinary Differential Equations

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9.1 Learning Objectives

At the end of this session, you will be able to do the following:

1. recognize and classify systems of ordinary differential equations.
2. solve linear first-order systems of ordinary differential equations.
3. represent higher-order ordinary differential equations as systems of first-order ordinary differential equations.
4. write linear approximations of nonlinear systems of ordinary differential equations.
5. identify research problems where systems of differential equations can be used to model the system.

9.2 Definitions

System of Ordinary Differential Equations: an set of equations, each containing a function of independent variables and their derivatives. In this class, we will focus on linear first-order systems. Why? **Linear** (no products, powers, or higher order functions): Lots of important problems are linear. Also, linearized versions of nonlinear systems are used to analyze behavior near **equilibrium points** (points where derivatives are zero).

First-order (only first derivatives): First-order systems describe a lot of important phenomena and are easy to solve analytically and numerically. Higher order systems often can be reduced to first-order systems.

System (set of equations): Here we'll focus on systems where the number of variables is the same as the number of equations, e.g. here is an example of a 2x2 system:

$$x' = x + 3y$$

$$y' = x - y$$

9.3 Example: Lotka-Volterra Cycles

Lotka-Volterra cycles were the result of early efforts to model predator-prey interactions. The classic example used is the interaction between lynxes and snow hares. The interaction is modeled as a system of differential equations:

$$\frac{dL}{dt} = \alpha LR - \beta L \frac{dR}{dt} = \gamma R - \delta LR$$

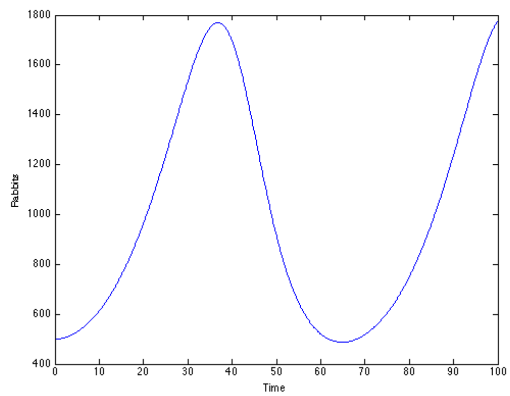
The four terms in this system are a combination of two terms describing predation (depending on both the number of lynx and the number of hares) and two terms describing the animal populations on their own (exponential growth of hares and decay of lynxes). Let's look for a stable point by setting both $\frac{dL}{dt}$ and $\frac{dR}{dt} = 0$ Ignoring the (trivial) case where L and R are zero (extinction), we wind up with

$$R = \frac{\beta}{\alpha} L = \frac{\gamma}{\delta}$$

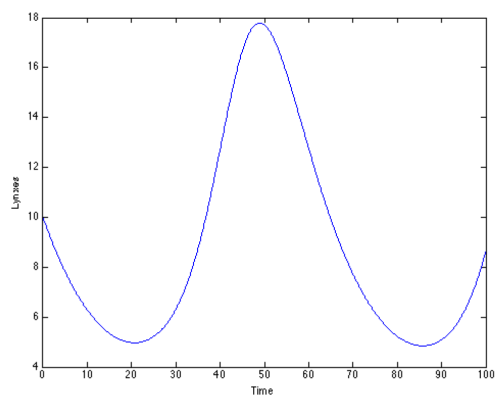
If the number of rabbits and lynxes were magically at these numbers right from the get go they would remain there forever. (If the model is correct.) If you initialize the model with slightly different values you'll see that the number of rabbits and lynxes oscillate around the stable point. That's what is shown below.

9.3.1 Lotka-Volterra code

```
%Lotka-Volterra Example
%Set parameters
a=0.1;
b=0.01;
c=0.0001;
d=0.1;
%Timestep values
dt=0.1;
%Initial conditions
L=zeros(1000,1);
R=zeros(1000,1);
L(1)=10; R(1)=500;
%Loop for some amount of time
it=1;
for i=1:1000
    dR=a*R(it) - b*R(it)*L(it);
    dL=c*R(it)*L(it)- d*L(it);
    R(it+1)=dR*dt+R(it);
    L(it+1)=dL*dt+L(it);
    it=it+1;
end
```



Rabbits
Time



Lynxes
Time

9.4 Writing a system of equations as a matrix equation

The system above can also be written as a matrix equation. For now, let's keep things more general and write down a first-order linear 2x2 system with arbitrary coefficients:

$$x' = ax + by$$

$$y' = cx + dy$$

This can be rewritten as a matrix equation:

$$\vec{x}\,' = A\vec{x}$$

where $\vec{x}$ is the vector $\begin{bmatrix} x \\ y \end{bmatrix}$ and A is the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Check for yourself that the this works! Also, try it out with a 3x3 system:

$$x' = x + y - z$$

$$y' = 2x - y$$

$$z' = 2y + z.$$

9.5 Solving a matrix equation: eigenvalues and eigenvectors

How do we solve the system $\vec{x}\,' = A\vec{x}$? Recall that solutions to a single first-order differential equation equation $x' = ax$ have the form $x = Ce^{at}$. For the matrix equation, let's try solutions of the form $\vec{x} = e^{\lambda t}\vec{v}$, where $\vec{v} = \begin{bmatrix} p \\ q \end{bmatrix}$, and p, q, and λ are unknown constants. Plugging in this solution, we get:

$$\vec{x}\,' = A\vec{x}$$

$$\lambda e^{\lambda t}\vec{v} = Ae^{\lambda t}\vec{v}$$

Canceling the exponential functions and switching the left and right sides, we find:

$$A\vec{v} = \lambda\vec{v}$$

$$(A - \lambda I)\vec{v} = 0$$

This equation has solutions for non-zero $\vec{v}$ when the determinant of $(A - \lambda I)$ is zero, as stated in the "characteristic equation":

$$det(A - \lambda I) = 0$$

The eigenvalues λ are the roots of the characteristic polynomial $det(A-\lambda I)$, and for each eigenvalue a corresponding eigenvector can be found by solving $(A - \lambda I)\vec{v} = 0$.

For example, for the 2x2 system in section 4,

$$(A - \lambda I) = \begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix}$$

and the characteristic equation is

$$det(A - \lambda I) = 0 = (a - \lambda)(d - \lambda) - cb = \lambda^2 - (a + d)\lambda + (ad - bc)$$

The general solution to the system is $\vec{x}_1 = \vec{v_1} e^{\lambda_1 t} + \vec{v_2} e^{\lambda_2 t}$, where $\lambda_1$and λ_2 are the roots of the characteristic equation, and $\vec{v_1}$ and $\vec{v_2}$ are the corresponding eigenvectors, found by solving $(A - \lambda I)\vec{v} = 0$.

Below, we'll go through some examples for cases where the eigenvalues are real and distinct, complex, or repeated.

9.5.1 Real, distinct eigenvalues

Real eigenvalues imply exponential growth (positive values) or decay (negative) along their respective eigenvectors.

Example

Find the solution to the system below, with initial conditions x(0)=1 and y(0)=0.

$$x' = -2x + y$$
$$y' = -4x + 3y$$

First, rewrite the system as a matrix equation $A\vec{x} = \vec{x}'$, where $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, $A = \begin{bmatrix} -2 & 1 \\ -4 & 3 \end{bmatrix}$, and the initial condition is $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

This equation has solutions when $det(A - \lambda I) = 0$:

$$det \begin{bmatrix} -2 - \lambda & 1 \\ -4 & 3 - \lambda \end{bmatrix} = (-2 - \lambda)(3 - \lambda) - (-4)(-3) = \lambda^2 - \lambda - 2 = 0$$

Factor to find the roots of the equation:

$$\lambda^2 - \lambda - 2 = (\lambda + 1)(\lambda - 2) = 0$$

The eigenvalues are $\lambda_1 = -1$, $\lambda_2 = 2$.

Now find the eigenvectors by solving $(A - \lambda I)\vec{v} = 0$. First, the eigenvector corresponding to $\lambda_1 = -1$:

$$\begin{bmatrix} -2-(-1) & 1 \\ -4 & 3-(-1) \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -4 & 4 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}$$

$$-p + q = 0$$
$$-4p + 4q = 0$$

This is true for any $p = q$, so $\vec{v}_1 = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. (Note: An easy choice is to set the first value to 1, then solve, understanding that solutions are any multiple of the vector you find. When reporting eigenvectors, it's common to normalize the eigenvectors so that the magnitude $\sqrt{p^2 + q^2} = 1$.)

Now let's find the other eigenvector, corresponding to $\lambda_2 = 2$:

$$\begin{bmatrix} -2-2 & 1 \\ -4 & 3-2 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} -4 & 1 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}$$

$$-4p + q = 0$$

This is true for any $p = \frac{1}{4}q$, so $\vec{v}_2 = c_2 \begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

We have the eigenvalues and eigenvectors, so we can write down the general solution:

$$\vec{x} = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2$$

$$\vec{x} = c_1 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

All that's left is to solve for the constants using the initial condition:

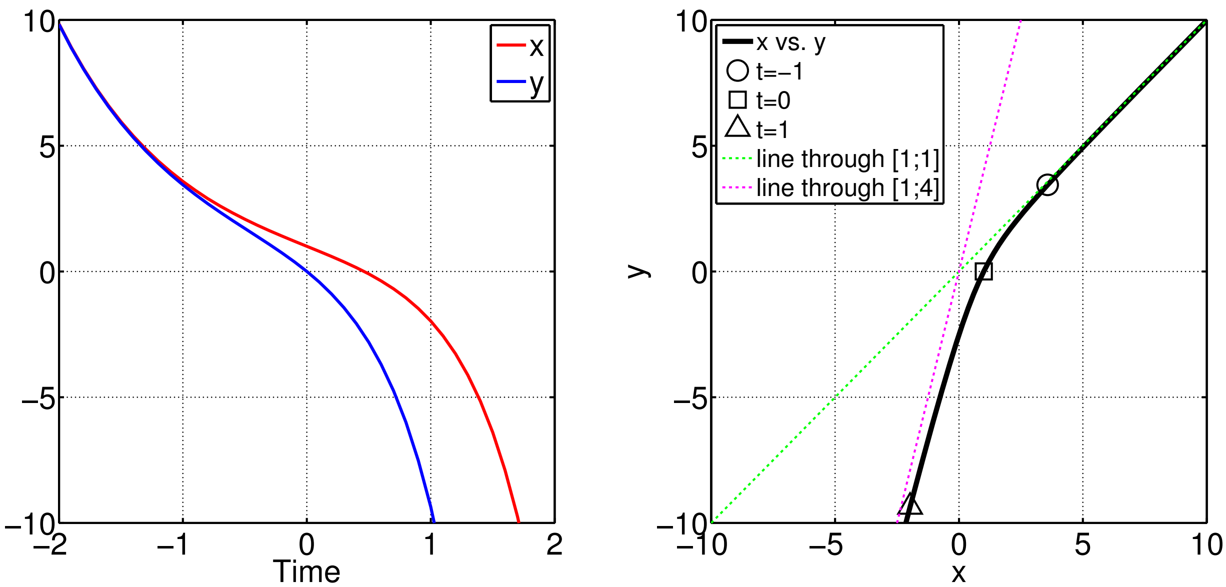
$$x(\vec{0}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

Solving by elimination, $c_1 = 4/3$ and $c_2 = -1/3$. The full solution is:

$$x = \frac{4}{3} e^{-t} - \frac{1}{3} e^{2t}$$

$$y = \frac{4}{3} e^{-t} - \frac{4}{3} e^{2t}$$

Interpretation: When t is a large negative number, the first term is big and the second is small: the solution is near the line through the first eigenvector. As t gets near zero, the two terms become comparable. When t is a large positive number, the second term is big and the first is small: the solution is near the line through the second eigenvector.



9.5.2 Complex eigenvalues

What happens when the eigenvalues are complex?

- Complex eigenvalues lead to "spiral" solutions (again, for continuous variables).
- If the real parts of the eigenvalues is positive, solutions are "unstable" (going to infinity with t approaching positive infinity).
- If the real parts of the eigenvalues are negative, solutions are "stable" (going to zero with t approaching positive infinity).
- If the real parts of the eigenvalues are zero, solutions are ellipses.
- The real and imagninary parts of the complex solution are each solutions to the system.

Example: Romeo & Juliet (from 18.03 lecture notes) Consider the system:

$$R' = J$$

$$J' = -\frac{17}{16}R + \frac{1}{2}J$$

R denotes Romeo's love for Juliet, and J denotes Juliet's love for Romeo. Here's the interpretation: "Romeo is a puppy dog. He has little self-awareness; the change in his feelings towards Juliet has nothing to do with how he himself feels at the moment; it is completely dependent on how she feels about him. Juliet is more complex. She has a healthy self awareness; if she loves him, that very fact causes her to love him more. On the other hand, if he seems to love her, she gets frightened and starts to love him less."

Let's rewrite the system into matrix form: $\vec{x}' = A\vec{x}$, with $\vec{x} = \begin{bmatrix} R \\ J \end{bmatrix}$ and $A = \begin{bmatrix} 0 & 1 \\ -\frac{17}{16} & \frac{1}{2} \end{bmatrix}$. Find the eigenvalues of A by the characeristic equation $det(A - \lambda I) = 0$ using the quadratic formula:

$$det \begin{bmatrix} 0-\lambda & 1 \\ -\frac{17}{16} & \frac{1}{2}-\lambda \end{bmatrix} = (-\lambda)(\frac{1}{2}-\lambda) - (-\frac{17}{16}) = \lambda^2 - \frac{1}{2}\lambda + \frac{17}{16} = 0$$

$$\lambda = \frac{1}{2}\left(\frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2 - 4\left(\frac{17}{16}\right)}\right) = \frac{1}{4} \pm i$$

Now find the eigenvectors by solving $(A - \lambda I)\vec{v} = 0$. First, the eigenvector corresponding to $\lambda_1 = \frac{1}{4} - i$:

$$\begin{bmatrix} 0-\frac{1}{4}+i & 1 \\ -\frac{17}{16} & \frac{1}{2}-\frac{1}{4}+i \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}$$

$$\left(-\frac{21}{16}+i\right)p + \left(\frac{5}{4}+i\right)q = 0$$

$$\frac{p}{q} = \frac{-\left(\frac{5}{4}+i\right)}{\left(-\frac{21}{16}+i\right)} = \left(\frac{4}{17} + \frac{16}{17}i\right)$$

So, we've found the first eigenvector:

$$\vec{v}_1 = c_1 \begin{bmatrix} \left(\frac{4}{17}+\frac{16}{17}i\right) \\ 1 \end{bmatrix}$$

Using the same procedure, we find that the eigenvector corresponding to $\lambda_2 = \frac{1}{4} + i$ is:

$$\vec{v}_2 = c_2 \begin{bmatrix} \left(\frac{4}{17}-\frac{16}{17}i\right) \\ 1 \end{bmatrix}$$

We have the eigenvalues and eigenvectors, so we can write down the general solution:

$$\vec{x} = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$$

$$\vec{x} = c_1 e^{\left(\frac{1}{4}-i\right)t} \begin{bmatrix} \left(\frac{4}{17}+\frac{16}{17}i\right) \\ 1 \end{bmatrix} + c_2 e^{\left(\frac{1}{4}+i\right)t} \begin{bmatrix} \left(\frac{4}{17}-\frac{16}{17}i\right) \\ 1 \end{bmatrix}$$

Let's plot the solution an example initial condition: $R(0) = \frac{1}{17}$, $J(0) = \frac{1}{4}$:

$$\vec{x}(0) = \begin{bmatrix} \frac{1}{17} \\ \frac{1}{4} \end{bmatrix} = c_1 \begin{bmatrix} \left(\frac{4}{17}+\frac{16}{17}i\right) \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} \left(\frac{4}{17}-\frac{16}{17}i\right) \\ 1 \end{bmatrix}$$

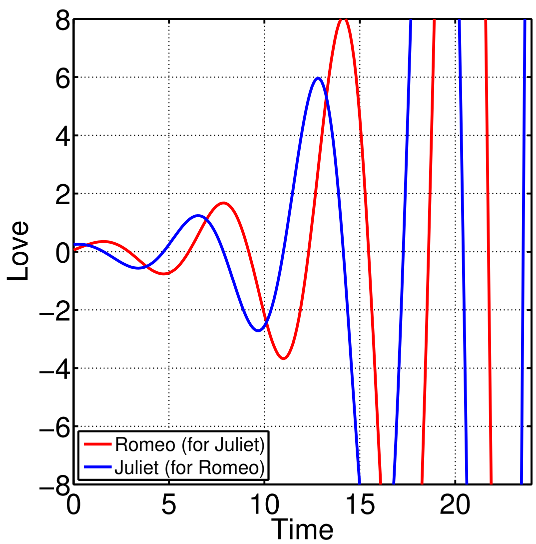
which is satisfied for $c_1 = c_2 = \frac{1}{8}$. The full solution is:

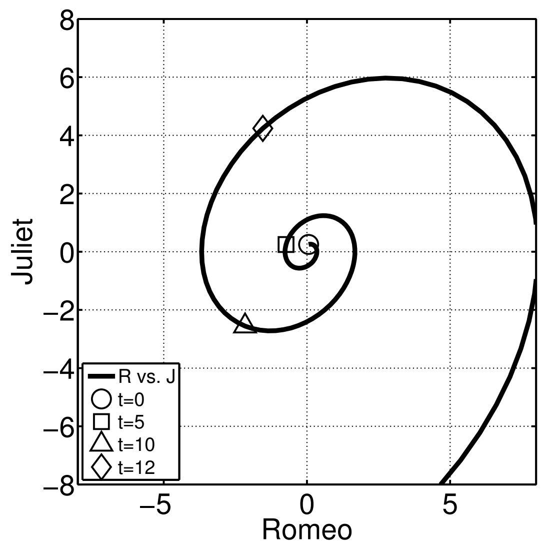
$$\vec{x}(0) = \begin{bmatrix} \frac{1}{17} \\ \frac{1}{4} \end{bmatrix} = \frac{1}{8} e^{\left(\frac{1}{4}-i\right)t} \begin{bmatrix} \left(\frac{4}{17} + \frac{16}{17} i\right) \\ 1 \end{bmatrix} + \frac{1}{8} e^{\left(\frac{1}{4}+i\right)t} \begin{bmatrix} \left(\frac{4}{17} - \frac{16}{17} i\right) \\ 1 \end{bmatrix}$$

and, after some simplification:

$$R(t) = \frac{1}{\sqrt{17}} e^{\frac{1}{4}t} \cos\left(t - \frac{\pi}{2} + \tan^{-1}\left(\frac{1}{4}\right)\right)$$
$$J(t) = \frac{1}{4} e^{\frac{1}{4}t} \cos(t)$$

Interpretation: Romeo and Juliet's love are oscillating (cosine), and the Romeo's love is about a quarter cycle $\left(\frac{\pi}{2}\right)$ behind Juliet's (with a small additional phase difference $\tan^{-1}\left(\frac{1}{4}\right)$). The amplitude of their love for each other is growing in time (like $e^{\frac{1}{4}t}$): each cycle, the amplitude spirals further and further out toward infinity.





9.5.3 Repeated eigenvalues

If the characteristic equation has a multiple root (e.g. for a 2x2 case, $det(A - \lambda I) = (\lambda - \lambda_1)^2$), the eigenvalue is called a "repeated" eigenvalue. One solution is found in the usual way, by solving $det(A - \lambda_1 I)\vec{v} = 0$, but the other solution (or additional solutions for larger systems) must be found in another way. Two cases are described below.

Example 1. The complete case.

In this case, there are two (or more for larger systems) linearly independent eigenvectors that correspond to the "repeated" eigenvalue. For example, solve $\vec{x}\,' = A\vec{x}$ with $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.

The characteristic equation has one repeated root:

$$det(A - \lambda I) = (\lambda - 2)^2$$

$$\lambda_1 = \lambda_2 = 2$$

Look for an eigenvector in the usual way:

$$(A - \lambda_1 I)\vec{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which says that ANY vector is an eigenvector. This is called the "complete" case. Things get more interesting for larger systems, e.g. the 3x3 case.

Example 2. The defective case.

If there is a repeated eigenvalue and all eigenvector solutions fall on a line (there is one independent eigenvector), the matrix is called "defective." (Sometimes this is also known as degenerative.) For example, solve $\vec{x}\,' = A\vec{x}$ with $A = \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix}$. The characteristic equation has one repeated root:

$$det(A - \lambda I) = \lambda^2 + 2\lambda + 1 = (\lambda + 1)^2$$

$$\lambda_1 = \lambda_2 = -1$$

One eigenvector is found in the usual way:

$$(A - \lambda_1 I)\vec{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

so $\vec{v}_1 = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and the corresponding solution is $\vec{x}_1 = e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

The trick to find the second solution is to solve:

$$(A - \lambda_1 I)\vec{w} = \vec{v}$$

and then try a solution of the form:

$$\vec{x}_2 = c_2 e^{\lambda_1 t}(t\vec{v} + \vec{w})$$

Solving for this case, $\vec{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, so the second solution is:

$$\vec{x}_2 = c_2 e^{-t} \left(t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$
$$\vec{x}_2 = c_2 e^{-t} \begin{bmatrix} t \\ t+1 \end{bmatrix}$$

So the general solution is:

$$\vec{x} = e^{-t} \left(c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} t \\ t+1 \end{bmatrix} \right)$$

9.6 Writing a higher-order ODE as a linear system

A higher-order differential equation often can be converted into a system of first-order equations that can be solved by the methods above.

Example 1. Convert $x'' - \frac{1}{2}x' + \frac{17}{16}x = 0$ into a first-order system. To convert this into a first-order equation, the trick is to define a new variable y such that $x' = y$. Differentiating, $x'' = y'$. We also know that $x'' = \frac{1}{2}x' - \frac{17}{16}x$ by rearranging the second-order equation above. Setting $y' = \frac{1}{2}x' - \frac{17}{16}x$ and substituting for x', we find $y' = -\frac{17}{16}x + \frac{1}{2}y$. The first-order system is the same one we solved in section 1.5:

$$x' = y$$
$$y' = -\frac{17}{16}x + \frac{1}{2}y$$

Example 2. Convert $x''' + x' + 2x = 0$ into a first-order system. Define $x' = y$, and note that $x''' = y'' = -x' - 2x = -y - 2x$. Now we need to apply the "trick" again and define another variable z such that $y' = z$. Noting that $y'' = z' = -y - 2x$, we now have the following first-order system:

$$x' = y$$
$$y' = z$$
$$z' = -2x - y$$

9.7 Write a nonlinear system of ODEs as a linear system

I will describe this for first-order, but it may of course be applied to higher-order systems. The easiest place to do this is around a fixed point of the system, where the derivatives are zero. Then we write the Jacobian of the original system (essentially a set of derivatives) which describe the way it behaves near that point.

We will use the Lotka-Voltera system as an example:

$$\frac{dL}{dt} = \alpha LR - \beta L \frac{dR}{dt} = \gamma R - \delta LR$$

There are two fixed points, one at extinction and the other at

$$R = \frac{\beta}{\alpha}$$

$$L = \frac{\gamma}{\delta}.$$

The Jacobian of (L', R') is that of their functions, $(\alpha LR - \beta L, \gamma R - \delta LR)$, which is a matrix with the derivative of each function with respect to each variable:

$$\begin{bmatrix} \frac{d}{dL}(\alpha LR - \beta L) & \frac{d}{dR}(\alpha LR - \beta L) \\ \frac{d}{dL}(\gamma R - \delta LR) & \frac{d}{dR}(\gamma R - \delta LR) \end{bmatrix}$$

$$= \begin{bmatrix} \alpha R - \beta & \alpha L \\ -\delta L & \gamma - \delta L \end{bmatrix}.$$

For the non-zero fixed point, this is:

$$\begin{bmatrix} 0 & \alpha\gamma/\delta \\ -\gamma & 0 \end{bmatrix}$$

For the all-zero fixed point, this is:

$$\begin{bmatrix} -\beta & 0 \\ 0 & \gamma \end{bmatrix}$$

These are the matrices of the system close to the fixed points. As it happens, near the origin the system is decoupled. Both systems can be solved.