MIT-WHOI Joint Program Summer Math Review	**Summer 2014**

Lecture 8: Ordinary Differential Equations

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8.1 Learning Objectives

At the end of this session, you will be able to do the following:

1. recognize and classify ordinary differential equations.
2. solve linear first-order ordinary differential equations.
3. solve constant-coefficient linear second-order differential equations.
4. identify research problems where differential equations can be used to model the system.

8.2 Definitions

Ordinary Differential Equation (ODE): an equation containing a function of one independent variable and its derivatives.

The general definition for an ODE is

$$x^{(n)} = F(t, x, x', ..., x^{(n-1)})$$

Where F is a given function of x,t, and derivatives of x. $x^{(n)}$ is the n-th derivative of the function x. This is an explicit ODE of **order** n.

F can also be given implicitly as follows:

$$F(t, x, x', ..., x^{(n)}) = 0$$

8.2.1 Example 1

A classic example of an ordinary differential equation is Newton's second law of motion $F = ma$.

Which can be written as: $m\frac{d^2x(t)}{dt^2} = F(x(t))$. Eqivalently: $F = mx''(t)$, $F = mx^{(2)}(t)$, $F = m\ddot{x}(t)$.

Here a, acceleration, is written as the second time derivative of position.

This is a second order ordinary differential equation where one aims to solve for x as a function of t.

8.2.2 More Definitions

Autonomous Differential Equation: A differential equation that doesn't depend on t (independent variable).

Linear DE: A differential equation is linear if F can be written as a linear combination of the derivatives of x.

Mathematically that looks like this:

$x^n = \sum_{k=0}^{n-1} a_k x^k + r(t)$

Where a_k is some coefficient and $r(t)$ is the source/sink term.

These linear differential equations have different names and solution methods depending on what your $r(t)$ is.

if $r(t) = 0$ then the equation is called **homogeneous**.

if $r(t) \neq 0$ then the equation is **nonhomogeneous**.

8.2.3 Classification Practice

Determine which of the following are ODEs. For each that is, note its order and determine whether it is any or all of: linear, autonomous, homogeneous, nonhomogeneous.

1. $$\frac{dy}{dt} + y = 0$$

2. $$y = x$$

3. $$\frac{dx}{dt} + x^2 = t$$

4. $$m\ddot{x} + b\dot{x} + kx = 0$$

5. $$\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial t^2} = 0$$

6. $$y' + t^2 y + t^3 = 0$$

8.3 Solving Differential Equations

8.3.1 Separable Differential Equations

There are several types of separable differential equations, but they all generally are solved by grouping t with functions and derivatives of t and doing the same for x.

Example 1. Here is an example of a differential equation that is separable in t:

$\frac{dx}{dt} = F(t)$

Where $F(t)$ is some function of t. Here is how this equation would be solved with $F(t) = t^2 + 5$:

$\frac{dx}{dt} = t^2 + 5$

$dx = (t^2 + 5)dt$

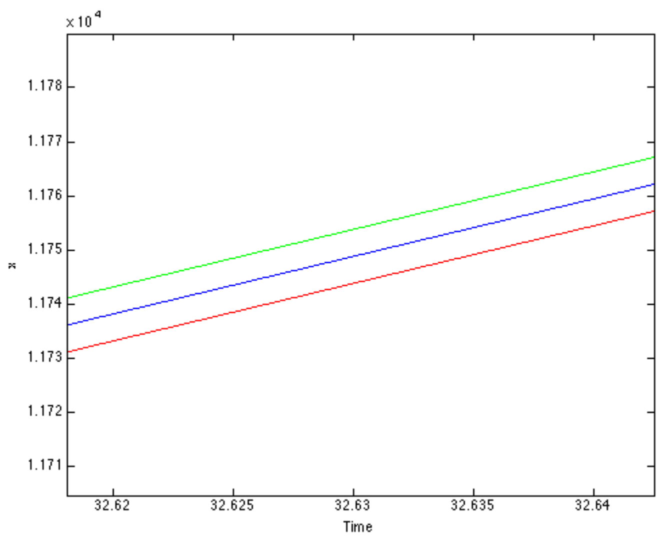
$\int dx = \int (t^2 + 5)dt$

After integrating you wind up with

$x = \frac{t^3}{3} + 5t + C$

Where C is an integration constant that requires an initial condition to resolve. This solution creates what's known as a **family of solutions** where different values of C create different solutions. An **initial condition** or **boundary condition** would result in a unique solution. From an initial condition, the statement $x(t = 0) = X$, you can find $C = X$. Boundary conditions are used when the independent variable represents space.

Family of Solutions (C=0, C=5, C=10)



Another form of separable differential equations is:

$$P_1(t)Q_1(x) + P_2(t)Q_2(x)\frac{dx}{dt} = 0$$

Where $P_1(t)$, $P_2(t)$, $Q_1(x)$, and $Q_2(x)$ are functions of t and x respectively.

To solve this form of separable ODE you first divide through by P_2Q_1 and then separate and integrate.

Example 2.

Let's take $P_1 = t^2$, $P_2 = t^4$, $Q_1 = x$, and $Q_2 = x^4$

The equation now looks like this:

$$t^2x + t^4x^4\frac{dx}{dt} = 0$$

$P_2Q_1 = t^4x$ so dividing through gives

$$\frac{1}{t^2} + x^3\frac{dx}{dt} = 0.$$

Separating (by moving one term to the other side and multiplying by dt) gives

$x^3dx = -\frac{dt}{t^2}$.

Then we integrate:

$\int x^3dx = \int -\frac{dt}{t^2}$, which gives

$\frac{x^4}{4} = \frac{1}{t} + C$.

Which again without an initial value gives you a family of solutions. Here, the 'initial value' must be from some time other than zero, as the solution is not defined there.

8.3.1.1 Practice

Solve:

1.

$$\frac{dy}{dt} + y = 0$$
$$y(0) = 5.$$

2.

$$tx' = t^2x^{-1}$$
$$x(1) = 1.$$

8.3.2 The Integrating Factor

The integrating factor solution to first order, linear, nonhomogeneous ODEs with function coefficients is a popular solution taught in most differential equations courses and comes up surprisingly often.

The differential equation must be in the form or be able to be put into the form:

$$\frac{dx}{dt} + P(t)x = Q(t)$$

To solve this you apply a formula involving something called the integrating factor. The integrating factor is

$$e^{\int P(t)dt}.$$

Multiplying the equation by this factor gives:

$$e^{\int P(t)dt}\frac{dx}{dt} + P(t)e^{\int P(t)dt}x = e^{\int P(t)dt}Q(t) = \frac{d}{dt}(e^{\int P(t)dt}x),$$

by recognizing the chain rule. Then the equation can be simply integrated. The general solution looks like:

$$x = e^{-\int P(t)dt}\int Q(t)e^{\int P(t)dt}dt + Ce^{-\int P(t)dt}$$

Example:

$P(t) = \frac{2}{t}$, $Q(t) = 5t$

$$\frac{dx}{dt} + \frac{2}{t}x = 5t$$

Your integrating factor is then $e^{\int \frac{2}{t}dt}$ which works out to t^2.

Plugging into the general solution you get:

$$x = \frac{1}{t^2}\int 5t^3 dt + \frac{C}{t^2}.$$

Finally, after evaluating the integral you get

$$x = \frac{5t^2}{4} + \frac{C}{t^2}.$$

Note: if $Q(t) = 0$ then the equation is homogeneous and the solution is simply

$$x = \frac{C}{e^{\int P(t)dt}}$$

8.3.2.1 Practice

Solve:

1.
$$\begin{aligned} \frac{dy}{dt} + t^2 y &= t^2 \\ y(0) &= 5. \end{aligned}$$

2.
$$\begin{aligned} tx' &= x + t \\ x(1) &= 1. \end{aligned}$$

8.3.3 Solving Second Order Homogeneous and Nonhomogeneous ODEs

This method works for linear second-order nonhomogeneous linear differential equations with constant coefficients. Along the way you solve for the solution to the homogeneous equation, called the **complimentary function**. The equation to be solved will be of the form

$$ax'' + bx' + cx = G(t)$$

and the complimentary equation is

$$ax'' + bx' + cx = 0.$$

For equations of this form the general solution can be stated as follows:

$$x(t) = x_p(t) + x_c(t)$$

Where x_p is the particular solution to the nonhomogeneous equation and x_c is the solution to the complementary equation. This solution method depends on the **superposition principle**, which states that for linear functions, the output of a sum of inputs is equal to the sum of outputs of those inputs: $f(a + b) = f(a) + f(b)$. (There's a proof in Stewart Calculus if you're into that kind of thing.)

8.3.3.1 Auxiliary Equation

To solve the complementary equation, which is 2nd-order, homogeneous, and linear with constant coefficients, we form the **auxiliary** or **characteristic equation**. This equation allows us to find a solution to the complementary equation as the sum of two exponentials. All second-order equations must have two solutions, or one with two undetermined coefficients, to allow initial conditions on the function and it's first derivative.

Example Let the complimentary equation we are solving be

$$mx'' + bx' + kx = 0.$$

Then suppose $x = Ae^{rt}$, and plug that in:

$$mr^2Ae^{rt} + brAe^{rt} + kAe^{rt} = 0,$$
$$mr^2 + br + k = 0$$

where we have divided by x, is the characteristic equation. It is quadratic in r. The solutions give r_1 and r_2, and the full solution is then:

$$x = C_1e^{r_1t} + C_2e^{r_2t},$$

where C_1 and C_2 would be determined by initial conditions.

The possibilities for the complementary equation are as follows:

r_1,r_2 are real and distinct, then the general solution is $x = C_1e^{r_1t} + C_2e^{r_2t}$

$r_1 = r_2 = r$ then $x = C_1e^{rt} + C_2te^{rt}$

r_1, r_2 are complex $(\alpha \pm i\beta)$ then:

$x = e^{\alpha t}(C_1 \cos(\beta t) + C_2 \sin(\beta t))$

8.3.3.2 Method of Undetermined Coeffiecients

The first method we'll look at is the method of undetermined coefficients. In this method you make the assumption that the particular solution is a polynomial of the same degree as $G(t)$.

Here's an example to show what the process looks like: **Example**

$$x'' + x' - 2x = t^2$$

So we'll start by solving the complementary equation:

$$r^2 + r - 2 = (r - 1)(r + 2) = 0$$

This gives us a $x_c = c_1e^t + c_2e^{-2t}$

Now we look for x_p of the form:

$$x_p = At^2 + Bt + C$$

(Because $G(t)$ is a second degree polynomial)

With x_p of that form, $x_p' = 2At + B$ and $x + p'' = 2A$. Now substitute these into the solution and solve for the coefficients A, B, and C.

$$(2A) + (2At + B) - 2(At^2 + Bt + C) = t^2 - 2At^2 + (2A - 2B)t + (2A + B - 2C) = t^2$$

Now you solve the system to make the coefficients equal to $G(x)$

$-2A = 1$

$2A - 2B = 0$

$2A + B - 2C = 0$

This gives us $A = \frac{-1}{2}$, $B = \frac{-1}{2}$, and $C = \frac{-3}{4}$

Putting it all together:

$$x = x_c + x_p = c_1 e^t + c_2 e^{-2t} - \frac{1}{2}t^2 - \frac{1}{2}t - \frac{3}{4}$$

Note: If $G(t)$ is an exponential you guess $x_p(t) = Ae^{kt}$; if G is a trigonometric function, you guess the sum of that function and a derivative or two, etc. However, for trigonometric forms, the next method is often better.

8.3.3.3 Method of Variation of Parameters

In this method you start off by taking the general solution $x(t) = C_1 x_1(t) + C_2 x_2(t)$ and replace the constants by arbitrary functions $u_1(t)$ and $u_2(t)$.

Now we look for a particular solution that looks as follows:

$$x_p(t) = u_1(t)x_1(t) + u_2(t)x_2(t)$$

Now you differentiate the equation above and wind up with:

$$x_p' = (u_1' x_1 + u_2' x_2) + (u_1 x_1' + u_2 x_2')$$

Because these functions are arbitrary we can impose conditions on them to make our lives easier.

The condition commonly imposed is:

$$u_1' x_1 + u_2' x_2 = 0$$

It follows that:

$$x_p'' = u_1' x_1' + u_2' x_2' + u_1 x_1'' + u_2 x_2''$$

Then you can substitute all that into the differential equation. (We'll skip that because you'll see in the example)

Because both x_1 and x_2 are solutions to the complementary equation the end result is that

$$a(u_1' x_1' + u_2' x_2') = G$$

Example Solve the equation $x'' + x = \tan(t)$ from $0 < t < \frac{\pi}{2}$

The auxiliary equation is $r^2 + 1 = 0$

Which has the roots $r = \pm i$

So using the variation of parameters method we want a solution of the form:

$$x_p = u_1(t)\sin(t) + u_2(t)\cos(t)$$

Taking the derivative you wind up with:

$$x_p' = (u_1' \sin(t) + u_2' \cos(t)) + (u_1 \cos(t) - u_2 \sin(t))$$

Then we impose our condition:

$$u_1' \sin(t) + u_2' \cos(t) = 0$$

Then taking the second derivative:

$$x_p'' = u_1' \cos(t) - u_2' \sin(t) - u_1 \sin(t) - u_2 \cos(t)$$

Then adding x_p'' and x_p together (because the original equation was $x'' + x = tan(t)$) you wind up with:

$$u_1' \cos(t) - u_2' \sin(t) = \tan(t)$$

Our condition gives us:

$$u_2' = u_1' \frac{\sin(t)}{\cos(t)}$$

Substituting this into the x_p equation yields:

$$u_1'(\sin^2(t) + \cos^2(t)) = \cos(t) \tan(t)$$

Which gives: $u_1' = \sin t$ and $u_1 = -\cos t$

Plugging back into u_2' you get:

$$u_2' = -\frac{\sin t}{\cos t} u_1' = -\frac{\sin^2 t}{\cos t} = \frac{\cos^2 t - 1}{\cos t} = \cos t - \sec t$$

Then $u_2 = \sin t - \ln(\sec t + \tan t)$

Note that $\sec t + \tan t$ is positive from $0 < t < \frac{\pi}{2}$

so the particular solution is:

$$\begin{aligned} x_p &= -\cos(t) \sin(t) + [\sin(t) - \ln(\sec(t) + \tan(t))] \cos(t) \\ &= -\cos(t) \ln(\sec(t) + \tan(t)) \end{aligned}$$

and the general solution is:

$$x(t) = C_1 \sin(t) + C_2 \cos(t) - \cos(t) \ln(\sec(t) + \tan(t))$$

8.3.3.4 Practice

Solve using the method of undetermined coefficients:

$$x'' - x' + 2x = t^3.$$

Set up using variation of parameters:

$$x'' - x' + 2x = \cos(t).$$

8.4 Applications

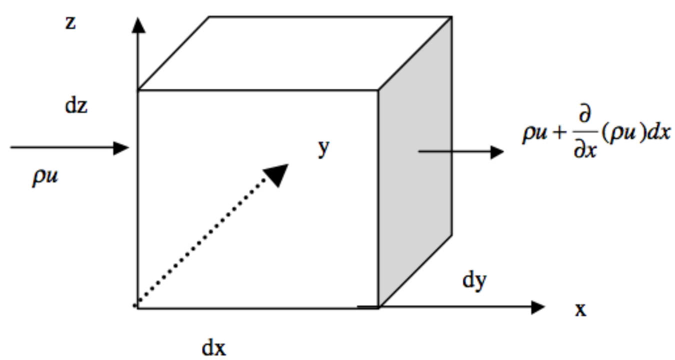
8.4.1 Control Volumes

A control volume is a space that you choose yourself for the purposes of a given problem. Its shape is arbitrary. Control volumes can either be moving or fixed. Control volumes are big in fluid mechanics, chemistry/chemical engineering, and environmental sciences. A good example of a readily selectable control volume is a stretch of river where you are interested in how the chemical and biological properties change.

8.4.2 Conservation of Mass

Using the concept of control volumes and doing some calculus you can derive conservation laws. The derivation works for any scalar quantity but the one you'll likely see most often is conservation of fluid mass.

To start imagine a stationary cube in a moving fluid. Where the fluid is flowing in the x direction with velocity u, the y direction with velocity v, and the z direction with velocity w. Your control volume (cube) has infinitesimal dimensions of δx, δy, and δz.



(Image from Pedlosky notes for 12.800)

Each face of the cube has an area ($\delta x\delta y$, $\delta x\delta z$, etc.) So the flux of fluid mass across the infinitesimal cube is the mass flow rate multiplied by the area it is crossing.

In the x-direction that looks like:

$$\rho u\delta y\delta z$$

As the fluid moves through the box is is possible for the fluid velocity or density to change by some amount

$$\frac{\partial\rho u}{\partial x}\delta x$$

And because you cannot randomly generate or lose mass the balance in the x-direction is

$$\frac{\partial \rho u}{\partial t} dv = \rho u - (\rho u + \frac{\partial \rho u}{\partial x} \delta x)$$

Now to get the flux you multiply through by $\delta y \delta z$. Which gives

$$\frac{\partial \rho u}{\partial x} \delta x \delta y \delta z = \frac{\partial \rho u}{\partial t} dv$$

Recognizing $\delta x \delta y \delta z = dv$ divide through by dv and you are left with

$$\frac{\partial \rho u}{\partial t} = \frac{\partial \rho u}{\partial x}$$

Doing the same in all directions (y and z) you get the conservation of mass equation (often called the continuity equation)

$$\frac{\partial \rho \vec{u}}{\partial t} = \mathrm{div}(\rho \vec{u})$$

Where div $= (\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z})$

8.4.3 Box Model and Example

Conservation laws often come into play in box models. Box models are popular in environmental fate and transport applications where you are interested in the change in concentration within a certain area with time.

Now, for example, take a perfectly mixed lake. Imagine that there is a large rainfall event that causes a large amount of agricultural run off. The rain causes a mass of 500kg of fertilizer to enter a nearby pond of volume 1000 m^3. (Here comes the cop out:) The fertilizer is instantly mixed across the entire volume of the pond and the only entrance and exit points are creeks flowing with $Q = 0.1ms^{-1}$.

The set up of this problem allows us to write and solve a nice differential equation for it.

Initially the concentration in the lake is $C_0 = 0.5\frac{kg}{m^3}$

$$V\frac{dc}{dt} = C_{in}Q_{in} - C_{out}Q_{out}$$

In a perfectly mixed environment $C_{out} = C$ and in this case we are assuming $C_{in} = 0$

That leaves:

$$V\frac{dc}{dt} = CQ_{out}.$$

Separating you wind up with:

$$V\frac{dc}{C} = Q_{out}dt.$$

Which integrates to:

$$ln(C) = V^{-1}Q_{out}t + A.$$

Rearranging and taking the exponent you wind up with:

$$C = e^{V^{-1}(Q_{out})t+A},$$

which is the same as:

$$C = Ae^{V^{-1}Q_{out}t}.$$

At time $t = 0$ the concentration was C_0 so plugging that in you wind up with:

$$C = C_0e^{V^{-1}Q_{out}t}$$

Which probably surprises no one.

8.4.4 Practice

Set up the following:

1. **Radioactive Decay** Suppose a radioactive element spills into a lake during the spring runoff. The lake now has $1000m^3$ of water and 1000mol of an isotope with a half life of 1000 days. There are no streams into the lake, but one out with a flow of $10m^3$ per day. The isotope also settles with sediments to the bottom of the lake at a rate of 0.1 mol/day. Assume that the lake is at all times well mixed. Write a differential equation for much of the isotope is in the lake over time. Write one for how much is in the sediments at the bottom. What solution method would you use for these problems?

2. **Fish Population** Suppose initially you have 1 kg of small fish per cubic meter, on average, in the mid-Atlantic Bight. Imagine that the fish like to be evenly spaced, and larger fish are more intimidating, so an average density is useful. Now, the fish grow (in size and through reproduction) as a function of the available zooplankton. Let the growth of the fish be 0.1 times a zooplankton index, $\sin(2\pi t/365)$, per fish density. Fish die for many reasons, including age, fishing, and predation. Let's assume that's a constant rate of 0.1 per fish density. Write a differential equation for the density of fish in time. What solution method would you use for this problem?

8.5 Useful Resources

Wikipedia 'Ordinary Differential Equations' – 'Integrating Factor'

Stewart Calculus (My favourite calculus books. Pretty much every edition I've seen is good)

Hibbeler (More so for Engineers)

Kundu-Cohen (Sort Of) – Some other fluid mechanics books do a better job at explanations but MIT seems to like this one.

Miller and Wheeler (for discussion of Lotka-Volterra cycles and other cool ecological models)